

STABILITY OF TANGENT BUNDLE ON THE MODULI SPACE OF STABLE BUNDLES ON A CURVE

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ABSTRACT. In this paper, we prove that the tangent bundle of the moduli space $\mathcal{SU}_C(r, d)$ of stable bundles of rank r and of fixed determinant of degree d (such that $(r, d) = 1$), on a smooth projective curve C is always stable, in the sense of Mumford-Takemoto. This verifies a conjecture of Peternell and others, and is related to a conjectural existence of a Kähler-Einstein metric on Fano varieties with Picard number one.

1. INTRODUCTION

Suppose X is a compact Kähler manifold. The existence of a Kähler-Einstein metric on X has attracted wide interest since the conjecture of Calabi and work of Yau [Ya] appeared, in the study of complex manifolds. Aubin [Au] and Yau show the existence of a Kähler-Einstein metric whenever the canonical line bundle K_X is ample or trivial. The existence of a Kähler-Einstein metric when $-K_X$ is ample, i.e., when X is a Fano manifold, is an open problem. This has many interesting applications and are discussed by Tian in [Ti2]. Kobayashi [Kb] and Lübke [Lu] show that the existence of a Kähler-Einstein metric implies the stability of the tangent bundle, in the sense of Mumford and Takemoto. In particular, the tangent bundle T_X is stable when X is of general type. Since then the stability problem for Fano manifolds has brought a lot of attention.

The significant works of Hwang [Hw], Peternell-Wisniewski [Pe-Wi], Steffens [St], Subramanian [Su], Tian [Ti] (and the references therein), prove the stability result for certain Fano manifolds X . In most of these cases, the Betti number $b_2(X) = 1$. When $b_2(X) > 1$, some examples are known when the stability fails, see [Ti2, p.183]. Since then it was speculated (for instance, by Peternell [Pe2, p.14, Conjecture 5.2]) that the stability of the bundle T_X holds when X is a Fano manifold with $b_2(X) = 1$.

Suppose C is a smooth projective curve of genus g . The moduli space $\mathcal{SU}_C(r, d)$ of stable vector bundles of rank r and of fixed determinant of degree d , is a projective Fano manifold when r and d are coprime. Furthermore, the Picard number is one generated by the *determinant line bundle* L and the canonical class $K = L^{-2}$. In other words, the moduli space is of index 2 [Ra]. When the rank $r = 2$ and $g \geq 2$, Hwang [Hw2, Theorem 1], proved the stability of the tangent bundle on $\mathcal{SU}_C(2, 1)$.

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In this paper, we prove the stability of the tangent bundle, for higher rank smooth projective moduli spaces. More precisely,

Theorem 1.1. *Suppose $r \geq 3$ and d is an integer such that $(r, d) = 1$. If the genus $g(C) \geq 3$, then the tangent bundle on the moduli space $\mathcal{SU}_C(r, d)$ is always stable, in the sense of Mumford-Takemoto.*

We first give a proof when the rank r is even. The key point is to use the action of the finite automorphism group on the moduli space and use descent conditions for the determinant line bundle L . This is done in §2.2. In §3, we give a proof for all $r \geq 3$. We use the Hecke correspondence [BLS, p.206], which relates the moduli spaces $\mathcal{SU}_C(r, 1)$ and $\mathcal{SU}_C(r, 1 - h)$ for any $h, 0 < h < r$. The main point is to use the structure of this correspondence and prove that the stability of the tangent bundle of the respective moduli spaces is preserved under this correspondence.

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2. STABILITY OF THE COTANGENT BUNDLE FOR THE MODULI SPACE $\mathcal{SU}_C(r, \eta)$

We start with some preliminaries to fix notations and definitions we will use.

2.1. Preliminaries. Suppose X is a projective manifold of dimension n and L is an ample line bundle on X . Suppose E is any coherent sheaf on X of rank k and degree d with respect to L . In other words, the determinant $\wedge^k E$ has the intersection number $d := (c_1(\wedge^k E) \cdot c_1(L)^{n-1})$. The slope of E is defined to be $\mu(E) := \frac{d}{k}$. Mumford-Takemoto stability means that, for any coherent subsheaf $F \subset E$, we have the inequality:

$$\mu(F) < \mu(E).$$

In the proofs, we will need to look at sheaves on a singular variety but whose singular locus has high codimension. We note the below lemma, which we will use.

Lemma 2.1. *Suppose X is a projective variety and $U \subset X$ is an open smooth subset. Let $S := X - U$ be the closed subset and which has codimension at least two. Let E be a coherent sheaf on X . Then $c_1(E)$ and $\mu(E)$ are well defined.*

Proof. See [Ma, p.318-319]. The key point is that $U = X - S$ is smooth and the Chern class $c_1(E|_U)$ of the restriction of E on U is well-defined, using a locally free resolution of $E|_U$. The Weil divisor $c_1(E|_U)$ extends uniquely on X since $\text{codim}(S) \geq 2$. Hence $c_1(E)$ and $\mu(E)$ are well-defined on X . \square

2.2. Stability for the moduli space $\mathcal{SU}_C(r, d)$, when r is even. Suppose C is a smooth projective curve of genus g . Let $\mathcal{SU}_C(r, d)$ denote the moduli space of stable bundles of rank r with fixed determinant η (of degree d) on C . Let $N := \dim \mathcal{SU}_C(r, d)$.

We note that the Picard number of X is one and let L be the ample generator of $\text{Pic} X$. Also X is of index two, i.e., the canonical line bundle $K_X = L^{-2}$ ([Ra, Theorem 1, p.69]).

Since the dual of a stable bundle is again stable, it suffices to prove that the cotangent bundle Ω_X^1 of the Fano manifold X is stable.

We remark that the stability of the cotangent bundle is implied by the vanishing of some Hodge cohomologies twisted by appropriate powers of the ample class L . This can be seen as follows. Suppose $S \subset \Omega_X^1$ is a coherent subsheaf of rank s and $\wedge^s S = L^k$, for some integer k . The inclusion of sheaves gives a non-trivial section of $\Omega_X^s \otimes L^{-k}$. The stability of the cotangent bundle will hold if we have the following vanishing:

$$H^0(X, \Omega_X^s \otimes L^{-k}) = 0, \text{ for } 0 < s < N, \text{ and } k \geq s \cdot \frac{-2}{N}.$$

Since $K_X = L^{-2}$, the condition on the slope is

$$(1) \quad \frac{k}{s} \geq \frac{-2}{N}, \text{ i.e. } -k \leq \frac{s}{N} \cdot 2 < 2.$$

In this situation we note that the stability or semistability of Ω_X give the same above inequality $-k < 2$.

Lemma 2.2. *With notations as above, the only possibility for k is equal to -1 , i.e. $\det S = L^{-1}$, for a destabilizing subsheaf $S \subset \Omega_X$.*

Proof. We exclude the other values of k as follows:

Case $-k < 0$: By Akizuki-Nakano vanishing theorem [Ak-Na], we have

$$H^0(X, \Omega_X^s \otimes L^{-k}) = 0 \text{ for any } 0 < s < N.$$

Case $k = 0$: Since X is Fano and hence rationally connected, the required Hodge cohomology vanish [Ko, p.202].

Case $-k > 0$: The slope condition (1) gives the only possibility $-k = 1$.

□

We first exclude this possibility when the rank r is even.

We prove the following statement:

Theorem 2.3. *Let X denote the moduli space $\mathcal{SU}_C(r, d)$ when r and d are coprime. If r is even, the tangent bundle T_X of X , is semistable in the sense of Mumford-Takemoto.*

Proof. Suppose the sheaf Ω_X is not semistable.

We note that the finite subgroup of r -torsion points $J[r]$ of the Jacobian $Jac(C)$ acts non-trivially on the moduli space $X = \mathcal{SU}_C(r, d)$:

$$E \mapsto E \otimes \eta, \text{ for } \eta \in Pic^0(C) = Jac(C).$$

We will use the action of $J[r]$ on the sheaf of differentials and apply descent.

Consider the quotient morphism $\pi : X \rightarrow Y := X/J[r]$. The quotient variety Y is the moduli space of PGL_r -bundles of degree d . The morphism π is unramified outside a closed subset of codimension at least two, i.e., π is étale in codimension one ([BLS, p.214]).

This means that $\Omega_X^1 = \pi^* \Omega_Y^1$ on the largest open subset $X^o \subset X$, where π is étale. Here Ω_Y^1 is the sheaf of Kähler differentials. Since π is a quotient morphism, there is a non-trivial action of $J[r]$ on the cotangent bundle Ω_X^1 . In other words, Ω_X^1 is a $J[r]$ -invariant sheaf.

Suppose $F \subset \Omega_X^1$ is a maximal destabilising subsheaf. Then, by Lemma 2.2, the only possibility is $\det(F) = L^{-1}$. Furthermore, by uniqueness (see [Hu-Le, Lemma 1.3.5, p.17]), F is also a $J[r]$ -invariant subsheaf. Hence on X^0 where π is unramified, by descent, $F = \pi^* F'$, for some coherent sheaf F' on $\pi(X^0)$. Since $\text{codim}(Y - \pi(X^0)) \geq 2$, by Lemma 2.1, $\det(F')$ extends uniquely on Y . This implies that the natural morphism $h : \pi^* \det(F') \rightarrow \det(F)$ is an isomorphism on X^0 and extends on X . Indeed, since this morphism gives a nonzero section $\mathcal{O}_X \rightarrow \det(F) \otimes (\pi^* \det(F'))^{-1}$ which is nonvanishing on X^0 . Now we use the fact that $\text{codim}(X - X^0) \geq 2$, to extend the isomorphism h on X .

By [BLS, Proposition 10.1, p.205], if r is even then $\det F = L^{-1}$ does not descend, and we get a contradiction.

□

3. STABILITY OF THE TANGENT BUNDLE OF $\mathcal{SU}_C(r, d)$, FOR ALL $r \geq 3$

We recall the Hecke correspondence from [BLS, p.206], and assume $r \geq 3$ in the rest of the paper:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{q'} & M' \\ \downarrow q & & \\ M & & \end{array}$$

Here $M := \mathcal{SU}_C(r, \eta)$ and $M' := \mathcal{SU}_C(r, \eta')$, such that $\deg \eta = 1$ and $\deg \eta' = 1 - h$, for $0 < h < r$. We can choose h so that $1 - h$ and r are coprime. Hence M and M' are smooth projective moduli spaces of dimension $N := (r^2 - 1)(g - 1)$. Moreover they are Fano varieties of index 2. The family $\mathcal{P} \rightarrow M$ is a family of Grassmanian varieties $G(h, r)$

and $q' : \mathcal{P} \rightarrow M'$ is a rational map. The general fibre of q' is a Grassmanian $G(r-h, r)$. See proof of [BLS, Proof of Lemma 10.3]. Let $N' := \dim \mathcal{P}$.

As in the previous section, denote the ample generator of $\text{Pic}M$ by L and of $\text{Pic}M'$ by L' . Then we have [BLS, Lemma 10.3, p.207]:

$$(2) \quad K_{\mathcal{P}} = q^*L^{-1} \otimes q'^*L'^{-1}.$$

Recall from (1) that stability of Ω_X or T_X (here $(X, \mathcal{L}) := (M, L)$ or (M', L')) will hold if there does not exist a coherent subsheaf

$$\mathcal{F} \subset T_X$$

of rank $p \leq \frac{N}{2}$ and $\det(\mathcal{F}) = L$.

Consider the exact sequence:

$$(3) \quad 0 \rightarrow T_{\mathcal{P}/M} \rightarrow T_{\mathcal{P}} \rightarrow q^*T_M \rightarrow 0.$$

A subsheaf of T_M will be considered as a subsheaf of $T_{\mathcal{P}}$ via the exact sequence (3), in the proof of Proposition 3.7 and in Theorem 4.1, at least over pullback of the Grassmanian bundle over étale coverings $U_{\alpha} \rightarrow M$.

To make this more precise we explain it in the below subsection.

3.1. The étale local triviality of the Grassmanian bundle. Consider the Grassmanian bundle $\mathcal{P} \rightarrow M$. By étale local triviality, we mean that there exist étale morphisms $p_{\alpha} : U_{\alpha} \rightarrow M$ such that the pullback bundle

$$\mathcal{P}_{U_{\alpha}} := \mathcal{P} \times_M U_{\alpha} \rightarrow U_{\alpha}$$

is a Zariski trivial fibration and the images of p_{α} cover M , i.e., $\cup_{\alpha \in I} p_{\alpha}(U_{\alpha}) = M$, for some indexing set I .

In the lemma below we note that such a covering exists.

Lemma 3.1. *There exists étale open sets $p_{\alpha} : U_{\alpha} \rightarrow M$, satisfying $\cup_{\alpha \in I} p_{\alpha}(U_{\alpha}) = M$, such that the pullback bundle $\mathcal{P}_{U_{\alpha}} \rightarrow U_{\alpha}$ is a Zariski trivial fibration.*

Proof. We need to note that the formal deformations of a rational homogeneous variety Z are trivial. This is a consequence of the well-known Bott's vanishing theorem: $H^1(Z, T_Z) = 0$. Since a Grassmanian is a rational homogeneous variety, the assertion on étale local triviality follows from [Se, Proposition 2.6.10]. \square

Fix étale morphisms $U_{\alpha} \rightarrow M$, for $\alpha \in I$, an index set, such that $\sqcup_{\alpha} U_{\alpha} \rightarrow M$ is surjective and the pullback $\mathcal{P}_{U_{\alpha}} \rightarrow U_{\alpha}$ is a product $U_{\alpha} \times G \rightarrow U_{\alpha}$. Here G is any fibre of q . Similar trivialization also hold for the bundle $\mathcal{P} \rightarrow M'$.

Definition 3.2. *We denote $M_{et} := \sqcup_{\alpha \in I} U_{\alpha}$ and $\mathcal{P}_{et} := \sqcup_{\alpha \in I} \mathcal{P}_{U_{\alpha}}$.*

Since q' is a rational map, defined outside a codimension two subset, but whose general fibre is a Grassmanian G' , the above discussion also applies to this fibration.

Instead of M , we could consider étale coverings $\sqcup_{\alpha \in I'} U'_\alpha \rightarrow M'$ of M' and denote

$$M'_{et} := \sqcup_{\alpha \in I'} U'_\alpha,$$

$$\mathcal{P}'_{et} := \sqcup_{\alpha \in I'} \mathcal{P}_{U'_\alpha},$$

using Lemma 3.1.

3.2. Coherent sheaves on \mathcal{P}_{et} . Call a disjoint union of étale coverings $M_{et} = \sqcup_{\alpha \in I} U_\alpha$, an atlas of M . Then to such an atlas, one can associate a Q -variety [Mu2]. Furthermore, by [Gi, Proposition 9.2], there is a regular stack \mathcal{M} associated to this data such that M is its coarse moduli space, i.e. there is a projection:

$$p : \mathcal{M} \rightarrow M.$$

Similarly, we have an associated Q -variety (or stack) for the atlas $\{\mathcal{P}_{U_\alpha}\}$ of \mathcal{P} .

In our context, it is possible to avoid stacks, since the regular stack associated to the étale coverings is again M . However we use the terminology of stacks to have the below isomorphism of rational Picard groups, and also to prescribe coherent sheaves. Sheaves on \mathcal{P}_{et} and in particular line bundles on \mathcal{P}_{et} are defined by prescribing such objects on an atlas compatible with overlaps ([Mu2, p.277]). In particular we have:

Lemma 3.3. *There is an isomorphism*

$$Pic(\mathcal{P}_{et}) \otimes \mathbb{Q} \simeq Pic(\mathcal{P}) \otimes \mathbb{Q}.$$

Similar isomorphisms hold over the sites M_{et} and M'_{et} .

Proof. See [Gi, p.195,0.5 Theorem]. □

In the below discussions, we will interchangeably use the terms étale site or atlas to mean that objects are defined over étale coverings and satisfying compatibility on overlaps, as in [Mu2, p.277].

Suppose \mathcal{G} is a sheaf on \mathcal{P} (respectively on M), then the pullback of \mathcal{G} along \mathcal{P}_{U_α} , for each $\alpha \in I$, gives a sheaf \mathcal{G}_{et} on \mathcal{P}_{et} (resp. M_{et}).

Consider the exact sequence (3), of tangent bundles on \mathcal{P} :

$$0 \rightarrow T_{\mathcal{P}/M} \rightarrow T_{\mathcal{P}} \rightarrow q^*T_M \rightarrow 0.$$

Lemma 3.4. *Suppose $\mathcal{F} \subset T_M$ is a coherent subsheaf and consider its pullback $q^*\mathcal{F} \subset q^*T_M$. Then it corresponds to a subsheaf $q^*\mathcal{F}_{et} \subset T_{\mathcal{P}_{et}}$ of the same rank.*

Proof. Using Lemma 3.1, we note that $\mathcal{P}_{U_\alpha} = U_\alpha \times G$. Hence the exact sequence (3) splits on \mathcal{P}_{U_α} :

$$0 \rightarrow T_G \rightarrow T_{\mathcal{P}_{U_\alpha}} = T_{U_\alpha} \oplus T_G \rightarrow T_{U_\alpha} \rightarrow 0.$$

Any coherent subsheaf $\mathcal{F} \subset T_M$ via pullback, corresponds to a subsheaf $\mathcal{F}_{U_\alpha} \subset T_{U_\alpha}$. Hence it is also a subsheaf of $T_{\mathcal{P}_{U_\alpha}}$ of the same rank. This is true for any $\alpha \in I$, and compatibility over overlaps holds because it holds on M_{et} . Hence it corresponds to a subsheaf $q^*\mathcal{F}_{et} \subset T_{\mathcal{P}_{et}}$.

□

Similar statement also holds for the fibration $q' : \mathcal{P} \rightarrow M'$.

Remark 3.5. *If there are two coverings $\sqcup_{\alpha \in I} \mathcal{P}_{U_\alpha} \rightarrow \mathcal{P}$ and $\sqcup_{\beta \in I'} \mathcal{P}_{U'_\beta} \rightarrow \mathcal{P}$, then we consider the common refinement*

$$\sqcup_{\alpha \in I, \beta \in I'} \mathcal{P}_{U_\alpha} \cap \mathcal{P}_{U'_\beta} \rightarrow \mathcal{P}.$$

Definition 3.6. *The common refinement corresponding to the étale local trivializations of $\mathcal{P} \rightarrow M$ and $\mathcal{P}' \rightarrow M'$ will be called the étale site $\mathcal{P}_{et} := \sqcup_{\alpha \in I, \beta \in I'} \mathcal{P}_{U_\alpha} \cap \mathcal{P}_{U'_\beta}$, in the later discussion.*

3.3. Stability is preserved under Hecke correspondence. In this subsection, we prove that the stability of the tangent bundle is preserved under Hecke correspondence. This will help us to conclude the stability of the tangent bundle for any $\mathcal{SU}_C(r, d)$, when $(r, d) = 1$, in the next section.

Recall the Hecke correspondence from the previous discussion, and $U \subset \mathcal{P}$ is the maximal open subset such that q' restricts to a morphism to M' .

Proposition 3.7. *Assume that $0 < h < r$ and $r \geq 3$ such that $(r, 1 - h) = 1$. A nonzero section of the group $H^0(U, \bigwedge^p T_{\mathcal{P}} \otimes q'^*L'^{-1})$ gives a nonzero section over $q(U)$:*

$$s : L \rightarrow \bigwedge^{p-m} T_M$$

Here $m := \dim G(h, r)$ and $m < p$.

Similarly, the statement also holds if we replace M by M' and M' by M .

Proof. Suppose there is a nonzero section $s' \in H^0(U, \bigwedge^p T_{\mathcal{P}} \otimes q'^*L'^{-1})$.

On U , consider the exact sequence (3). Then there is a finite filtration of each exterior power of $T_{\mathcal{P}}$ ([Ha, Ex 5.16 c), Chap.II]):

$$\bigwedge^p T_{\mathcal{P}} = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \mathcal{G}^2 \supset \dots \supset \mathcal{G}^{\binom{N}{p}+1} = 0$$

such that the i -th graded piece is

$$\frac{\mathcal{G}^i}{\mathcal{G}^{i+1}} = \left(\bigwedge^{p-i} q^*T_M \right) \otimes \left(\bigwedge^i T_{\mathcal{P}/M} \right).$$

Tensoring with $q'^*L'^{-1}$, we note that the section $s' \in H^0(U, \bigwedge^p T_{\mathcal{P}} \otimes q'^*L'^{-1})$ will be zero if it is zero in each (sub)quotient

$$H^0(U, \bigwedge^{p-i} q^*T_M \otimes \bigwedge^i T_{\mathcal{P}/M} \otimes q'^*L'^{-1})$$

for $i = 0, 1, \dots, p$. Hence it suffices to investigate the above cohomology groups. Let $m := \dim G(h, r)$. We claim that, for $i \leq p$ and except when $i = m < p$,

$$H^0(U, \bigwedge^{p-i} q^* T_M \otimes \bigwedge^i T_{\mathcal{P}/M} \otimes q'^* L'^{-1}) = 0.$$

case 1): $i = m \leq p$.

The cohomology group in this case is

$$H^0(U, \bigwedge^{p-i} q^* T_M \otimes K_{\mathcal{P}/M}^{-1} \otimes q'^* L'^{-1}) = H^0(U, \bigwedge^{p-i} q^* T_M \otimes q^* L^{-1}).$$

Indeed, using (2), we have

$$\begin{aligned} \bigwedge^m T_{\mathcal{P}/M} \otimes q'^* L'^{-1} &= K_{\mathcal{P}}^{-1} \otimes q^* K_M \otimes q'^* L'^{-1} \\ &= q^* L \otimes q'^* L' \otimes q^* L^{-2} \otimes q'^* L'^{-1} \\ &= q^* L^{-1}. \end{aligned}$$

If $i = p$ then the cohomology group is $H^0(U, q^* L^{-1}) = 0$, since L is ample. Otherwise, a nonzero section of the relevant cohomology group gives a nonzero section over $q(U)$:

$$s : L \rightarrow \bigwedge^{p-m} T_M.$$

case 2): $i < m$. Since the family $\mathcal{P} \rightarrow M$ is étale locally trivial, the pullback of the nonzero section s' on \mathcal{P}_{et} (see Definition 3.6), is a nonzero section in

$$H^0(M_{et}, \bigwedge^{p-i} T_M) \otimes H^0(G, \bigwedge^i T_G \otimes \mathcal{O}_G(-r)).$$

Here, we use the fact that $q'^* L'$ restricted on a fibre $G := G(h, r)$ of q is $\mathcal{O}_G(r)$ (see the proof of [BLS, Lemma 10.3]). Now notice that

$$\begin{aligned} H^0(G, \bigwedge^i T_G \otimes \mathcal{O}_G(-r)) &= H^0(G, \Omega_G^{m-i} \otimes K_G^{-1} \otimes \mathcal{O}_G(-r)) \\ &= H^0(G, \Omega_G^{m-i} \otimes \mathcal{O}(1)). \end{aligned}$$

But this group is zero, by [Le, Corollaire 1,(1), p.258] and we deduce the claim.

Similarly, since q' is also a Grassmanian bundle over M' , the above arguments also hold for the fibration $q' : \mathcal{P} \rightarrow M'$.

□

Corollary 3.8. *If the tangent bundle T_M of M is stable then the tangent bundle $T_{M'}$ of M' is also stable.*

Proof. Suppose $T_{M'}$ is not stable. Then, by (1), there exists a subsheaf $\mathcal{F}' \subset T_{M'}$ such that $\det(\mathcal{F}') = L'$ and rank of \mathcal{F}' is $p \leq \frac{N}{2}$. By Lemma 3.4, we consider the corresponding

subsheaf $q'^*\mathcal{F}' \subset T_{\mathcal{P}_{et}}$, on $U_{et} \subset \mathcal{P}_{et}$. This inclusion gives a nonzero morphism

$$q'^*\det(\mathcal{F}') \rightarrow \bigwedge^p T_{\mathcal{P}_{et}}.$$

This gives a nonzero section $s' \in H^0(U, \bigwedge^p T_{\mathcal{P}_{et}} \otimes q'^*L'^{-1})$.

By Proposition 3.7 (applied to each projection $\mathcal{P}_{U_\alpha} \rightarrow U_\alpha$), it corresponds to a nonzero section

$$s_\alpha : L \rightarrow \bigwedge^{p-m} T_{M_{et}}.$$

Here $m = \dim G(h, r)$. Since on any further refinement of the étale coverings, the sections s_α agree, so they patch and correspond to a nonzero morphism

$$s : L \rightarrow \bigwedge^{p-m} T_M$$

(see [Mi, p.64, 9, a)]) on $q(U)$.

By Lemma 2.1, since $\text{codim}(M - q(U)) \geq 2$, the below argument holds on M (take any extension of s on M).

We compute the slope of these sheaves. The slope of L is $\mu(L) = L.L^{N-1}$. The slope $\mu(\bigwedge^{p-m} T_M)$ is

$$\frac{\det(\bigwedge^{p-m} T_M).L^{N-1}}{\binom{N}{p-m}} = \frac{2 \cdot \binom{N-1}{p-m-1}}{\binom{N}{p-m}}.L^N$$

Claim: $\mu(L) > \mu(\bigwedge^{p-m} T_M)$.

We need to show that

$$\begin{aligned} 1 &> \frac{2 \cdot \binom{N-1}{p-m-1}}{\binom{N}{p-m}} = \frac{2 \cdot ((\binom{N}{p-m}) - \binom{N-1}{p-m})}{\binom{N}{p-m}} \\ &= 2 \cdot (1 - \frac{\binom{N-1}{p-m}}{\binom{N}{p-m}}) \\ &= 2 \cdot (\frac{p-m}{N}) \end{aligned}$$

Since $p-m < p \leq \frac{N}{2}$, this inequality always holds.

Since T_M is assumed to be stable, the exterior powers $\bigwedge^{p-i} T_M$, $i \geq 0$, are semistable [Ma]. Hence the above slope inequality gives a contradiction to the assumption.

□

4. STABILITY OF THE TANGENT BUNDLE T_M

In this section, we give a proof of our main result, using the Hecke correspondence and which preserves stability properties.

More precisely, we show:

Theorem 4.1. *The tangent bundle T_M of the moduli space $M = \mathcal{SU}_C(r, d)$ where $(r, d) = 1$ and $r \geq 3$ is always stable.*

Proof. Recall the Hecke correspondence and notations, from §3:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{q'} & M' \\ \downarrow q & & \\ M. & & \end{array}$$

(Here q' is a rational map and the pullback of sheaves from M' actually exist on an open subset $U \subset \mathcal{P}$ such that $\text{codim}(\mathcal{P} - U) \geq 2$. However for simplicity, we write \mathcal{P} instead of U below, since by Lemma 2.1, we can take any extension on \mathcal{P} and the arguments remain the same.)

By Corollary 3.8, we can assume that both T_M and $T_{M'}$ are not stable.

Suppose $\mathcal{F} \subset T_M$ (resp. $\mathcal{G} \subset T_{M'}$) is a coherent subsheaf destabilizing T_M (resp. $T_{M'}$). Then we have seen in §2.2 that we have the following only possibilities

$$(4) \quad \det(\mathcal{F}) = L, \det(\mathcal{G}) = L'$$

and

$$(5) \quad \text{rank}(\mathcal{F}) = p \leq \frac{N}{2}, \text{rank}(\mathcal{G}) = p' \leq \frac{N}{2}.$$

Consider the exact sequences

$$0 \rightarrow T_{\mathcal{P}/M} \rightarrow T_{\mathcal{P}} \rightarrow q^*T_M \rightarrow 0$$

and

$$0 \rightarrow T_{\mathcal{P}/M'} \rightarrow T_{\mathcal{P}} \rightarrow q'^*T_{M'} \rightarrow 0.$$

Using Lemma 3.4, $q^*\mathcal{F}$, $q'^*\mathcal{G}$ correspond to subsheaves of $T_{\mathcal{P}}$ of the same rank on the étale site \mathcal{P}_{et} . All the computations given below are over \mathcal{P}_{et} , M_{et} and M'_{et} .

We consider the subsheaves

$$q^*\mathcal{F} \cap q'^*\mathcal{G}, q^*\mathcal{F} + q'^*\mathcal{G} \subset T_{\mathcal{P}}.$$

Let $s := \text{rank}(q^*\mathcal{F} \cap q'^*\mathcal{G})$ and $s' := \text{rank}(q^*\mathcal{F} + q'^*\mathcal{G})$. Since $q^*\mathcal{F} \cap q'^*\mathcal{G} \subset q^*T_M$, $s \leq N$ and from (5), we deduce that $s' \leq N$. But $\text{rank } T_{\mathcal{P}} = N + \dim G$, here G is a fibre of q . Hence these are proper subsheaves of $T_{\mathcal{P}}$ and the ranks $s, s' < \text{rank}(T_{\mathcal{P}}) =: N'$.

Consider the exact sequence of sheaves:

$$(6) \quad 0 \rightarrow \mathcal{K} \rightarrow q^*\mathcal{F} \oplus q'^*\mathcal{G} \xrightarrow{\eta} q^*\mathcal{F} + q'^*\mathcal{G} \rightarrow 0$$

on \mathcal{P}_{et} . Here $\mathcal{K} := \text{kernel } (\eta)$. Also consider the exact sequence

$$(7) \quad 0 \rightarrow q^*\mathcal{F} \cap q'^*\mathcal{G} \rightarrow q^*\mathcal{F} \rightarrow T_{\mathcal{P}}/q'^*\mathcal{G} \rightarrow \mathcal{H} \rightarrow 0.$$

for some coherent sheaf \mathcal{H} .

To compute the determinants, we use the fact that the Picard group of \mathcal{P} is generated by $q^*\text{Pic}M$ and $\mathcal{O}_{\mathcal{P}}(1)$, and use Lemma 3.3 for the identification of $\text{Pic}(\mathcal{P}) \otimes \mathbb{Q}$ with $\text{Pic}(P_{et}) \otimes \mathbb{Q}$. (However note that $\text{Pic}(M)$ has no torsion).

Suppose the sheaf $\mathcal{K} = 0$. In this case we note that $\det(q^*\mathcal{F} + q'^*\mathcal{G}) = q^*L \otimes q'^*L'$. We deal with this case in the end. So we assume that $\mathcal{K} \neq 0$.

Suppose $\det(\mathcal{K}) = q^*L^a \otimes q'^*L'^b$, for some a, b , (see [Kd-Mu], for definition and functorial properties of the functor \det). The above exact sequence (6) and (4) give

$$\det(q^*\mathcal{F} + q'^*\mathcal{G}) = q^*L^{1-a} \otimes q'^*L'^{1-b}.$$

Now consider the morphism of sheaves:

$$(8) \quad \mathcal{K} \hookrightarrow q^*\mathcal{F} \oplus q'^*\mathcal{G} \rightarrow q^*\mathcal{F} \subset q^*T_M.$$

On an open subset of \mathcal{P} this is just the inclusion

$$q^*\mathcal{F} \cap q'^*\mathcal{G} \hookrightarrow q^*T_M.$$

Hence taking determinants in (8), we get a nonzero morphism

$$q^*L^a \otimes q'^*L'^b \rightarrow \bigwedge^s q^*T_M.$$

This gives a nonzero section of

$$H^0(\mathcal{P}_{et}, \bigwedge^s q^*T_M \otimes q^*L^{-a} \otimes q'^*L'^{-b}).$$

Since $T_{\mathcal{P}}$ splits on \mathcal{P}_{et} , we get a nonzero section in

$$H^0(M_{et}, \bigwedge^s T_M \otimes L^{-a}) \otimes H^0(G, q'^*L'^{-b}).$$

But $H^0(M_{et}, \bigwedge^s T_M \otimes L^{-a}) = H^0(M, \bigwedge^s T_M \otimes L^{-a}) = H^0(M, \Omega_M^{N-s} \otimes L^{2-a})$, (use $K_M = L^{-2}$). If $2-a \leq 0$, then by Kodaira-Akizuki-Nakano theorem and rational connectedness,

$$H^0(M, \Omega_M^{N-s} \otimes L^{2-a}) = 0.$$

If $b > 0$, then this is not possible since q'^*L' on G is $\mathcal{O}_G(r)$ (see proof of [BLS, Lemma 10.3]).

Hence $b \leq 0$ and $a \leq 1$.

Similarly, replacing $q^*\mathcal{F}$ by $q'^*\mathcal{G}$, if we consider the morphism $\mathcal{K} \rightarrow q'^*\mathcal{G}$ and the fibration $q' : \mathcal{P} \rightarrow M'$ we deduce that $a \leq 0$.

Now consider the subsheaf

$$(9) \quad q^*\mathcal{F} + q'^*\mathcal{G} \subset T_{\mathcal{P}}.$$

On the étale site \mathcal{P}_{et} we have a decomposition:

$$T_{\mathcal{P}} = T_M \oplus T_G.$$

Hence, taking determinants of above inclusion of sheaves in (9), we get a nonzero section in

$$\begin{aligned} H^0(\mathcal{P}_{et}, \bigwedge^{s'} T_{\mathcal{P}} \otimes q^* L^{a-1} \otimes q'^* L'^{b-1}) &= H^0(\mathcal{P}_{et}, \Omega_{\mathcal{P}}^{N'-s'} \otimes K_{\mathcal{P}}^{-1} \otimes q^* L^{a-1} \otimes q'^* L'^{b-1}) \\ &= H^0(\mathcal{P}_{et}, \Omega_{\mathcal{P}}^{N'-s'} \otimes q^* L^a \otimes q'^* L'^b). \end{aligned}$$

Using the splitting $\Omega_{\mathcal{P}} = \Omega_M \oplus \Omega_G$, this group is equal to

$$\bigoplus_{l+l'=N'-s'} H^0(M_{et}, \Omega_M^l \otimes L^a) \otimes H^0(G, \Omega_G^{l'} \otimes L^b).$$

Since $a \leq 0$, the groups $H^0(M_{et}, \Omega_M^l \otimes L^a) = H^0(M, \Omega_M^l \otimes L^a)$ are zero (by Kodaira-Akizuki-Nakano theorem and rational connectedness), except when $l = 0, a = 0$.

When $l = 0$, then $l' > 0$ and we have $b \leq 0$. Hence again by Kodaira-Akizuki-Nakano theorem and rational connectedness, $H^0(G, \Omega_G^{l'} \otimes L^b) = 0$.

Finally when $\mathcal{K} = 0$, then $a = 0, b = 0$, and this case is excluded by the above computations.

This completes the proof. \square

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